Linear Algebra

Introduction to Vectors[1]

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"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics."

Hermann Weyl 1885-1955, Mathematician and Philosopher

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Intended Learning Outcomes

By the end of this session you will be able to...

- Understand the basic concepts of linear algebra and vectors.
- Outline the rules governing operations on vectors.
- Investigate properties of vectors.
- Analysis examples of solving problems using vectors.

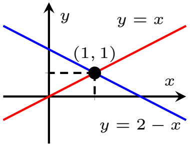
Question

What is linear algebra?

- Linear algebra arises from a need to solve **systems of linear equations**.

$$x + y = 2$$
$$x - y = 0$$

Algebraically



Geometrically

- Linear algebra plays an important role in many areas of pure and applied mathematics.
- Computers these days solve systems with thousands of linear equations every minute.

Please **scan** the barcode with your **phone** in order to take part in the class activity.

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Code: 84 44 8

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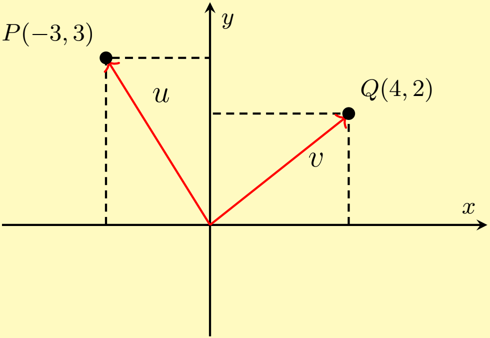
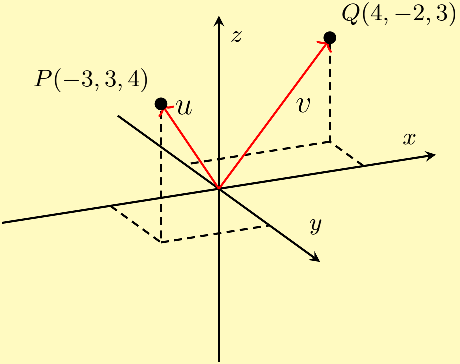
What is a vector?

Think about the 2-dimensional space $\mathbb{R}^2$



What is a vector?

Think about the 3-dimensional space $\mathbb{R}^3$



Geometry: Intuition

- Many physical quantities, such as temperature and speed, possess only **magnitude**.
- These quantities can be represented by **real numbers** and are called **scalars**.
- Vectors have **magnitude** and **direction**.
- They are represented by tuples, for example,

$$u = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}, \; v = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix}.$$

Vector Addition

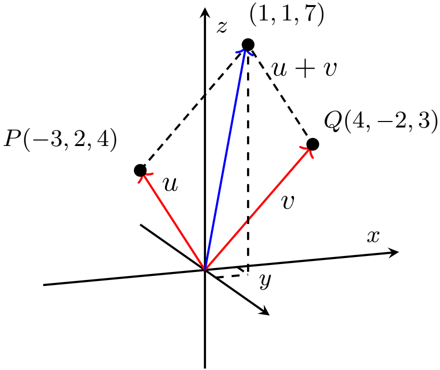
Algebraically the result of adding two vectors is **component-wise addition**. For example,

if

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \ b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix},$$

then

$$a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}.$$



Geometrically the result of adding two vectors is obtained by the **parallelogram law**.

Scalar Multiplication

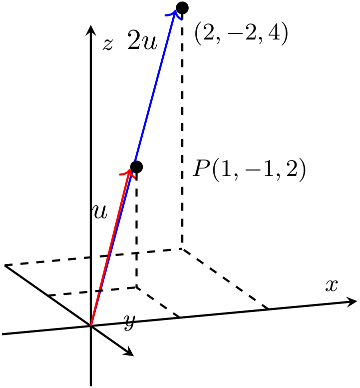
Algebraically the result of multiplying a vector by a scalar λ is **component-wise**. For example,

if

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

then

$$\lambda a = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{pmatrix}.$$



Geometrically the result of adding two vectors is obtained by **scaling the vector**, changing direction if $\lambda < 0$.

n-dimensional Euclidean Space

For a natural number n let $\mathcal{V} = \mathbb{R}^n$ with addition and scalar multiplication

$$(u_1, u_2, ..., u_n) + (v_1, v_2, ..., v_n) = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n)$$
$$\lambda(u_1, u_2, ..., u_n) = (\lambda u_1, \lambda u_2, ..., \lambda u_n),$$
$$\mathbf{0} = (0, 0, ..., 0).$$

In such case for $u = (u_1, u_2, ..., u_n)$ the vector $\widetilde{u}$ such that $u + \widetilde{u} = \mathbf{0}$ is give by

$$-u = (-u_1, -u_2, ..., -u_n).$$

Exercise

Let $u = (2, 4, -5, 1)$ and $v = (1, 2, 3, 4)$. Find

$$u + v, \; 3v, \; -v, \; 2u - 3v.$$

General Vectors in $\mathbb{R}^n$

Vectors is $\mathbb{R}^n$ form a set $\mathcal{V}$, with elements $u, v, w, \ldots$, together with addition $+$ and a scalar multiplication so that

$$u + v \in \mathcal{V} \text{ and } \lambda u \in \mathcal{V} \text{ for all } u, v \in \mathcal{V},\ \lambda \in \mathbb{R}.$$

In addition, for any $u, v, w \in \mathcal{V}$ and $\lambda, \mu \in \mathbb{R}$ the following **axioms** are satisfied.

Group Axioms

1. $u + v = v + u$
2. $(u + v) + w = u + (v + w)$
3. There exists $\mathbf{0} \in \mathcal{V}$ such that $u + \mathbf{0} = u$
4. There exists $\widetilde{u} \in \mathcal{V}$ such that $u + \widetilde{u} = \mathbf{0}$
   $\widetilde{u}$ is denoted by $-u$

Scalar Axioms

5. $\lambda(u + v) = \lambda u + \lambda v$
6. $(\lambda + \mu)u = \lambda u + \mu u$
7. $\lambda(\mu u) = (\lambda\mu)u$
8. $1u = u$

Exercise

Two vectors $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are equal if $u_i = v_i$ for every $i = 1, ..., n$. Find x, y, z so that

$$(x - y, x + z, z - 1) = (1, 2, 3).$$

Challenge Exercise (Function Spaces)

Let X be a set and $\mathrm{M}(X, \mathbb{R})$ the set of all functions $f : X \longrightarrow \mathbb{R}$ with addition and scalar multiplication

$$(f + g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda f(x).$$

Show $\mathrm{M}(X, \mathbb{R})$ satisfies the 8 axioms on slide 10.

What we did today...

- Linear algebra — History and applications
- Basics of vectors in $\mathbb{R}^n$ — Addition and scalar multiplication
- Vector axioms — Euqality of vectors and examples
- Exercises — Go through them and we check solutions
- Next time — Dot product and length of vectors

See You Next Time

Please Do Not Forget To

- Ask any **questions** now or through my contact details.
- Drop me **comments** and **feedback** relating to any aspects of the course.
- **My office hours** are on Mondays 15:00-16:00 & Fridays 11:00-12:00.
  Alternative: open door policy or email to make an appointment.

Thank You!