MATH1185 Vectors and Matrices[1]

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"This, therefore, is Mathematics; she reminds you of the invisible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas..."

PROCLUS 414 - 485 AD

[1] Use these notes in conjunction with R and Python demos.

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Contents

Lecture Contents

Aims and Outcomes

This module aims to introduce students to fundamental ideas and techniques in Linear Algebra. The primary objects concerned are vectors and matrices, and the module aims to make students proficient in performing operations involving these objects, which play central roles in higher level mathematics.

Learning Outcomes: on successful completion of this module a student will be able to:

1. Demonstrate fluency in vector algebra
2. Demonstrate fluency in matrix algebra
3. Apply fundamental techniques in linear algebra to solve appropriate mathematical problems

Topics to be Covered...

1. Sets, Functions, and Writing Mathematically
2. Linear Equations and Vectors I
3. Linear Equations and Vectors II
4. Matrix Algebra I
5. Matrix Algebra II
6. Determinants and Properties
7. Vector Spaces, Bases, and Dimension
8. Eigenvalues and Eigenvectors
9. Inner Product, Length, and Orthogonality

Assessment, Suggested Reading List and References

Assessment

Coursework 1, weight 50% due 18/11/2021
Coursework 2, weight 50% due 16/12/2021

For reading list see Lay et al. (2015); Strang (2016); Houston (2009).

Houston, K.
2009. *How to Think Like a Mathematician: A Companion to Undergraduate Mathematics*. Cambridge University Press.

Lay, D., S. Lay, and J. McDonald
2015. *Linear Algebra and Its Applications*. Pearson Education.

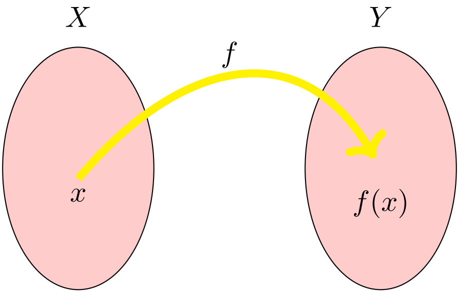
Strang, G.
2016. *Introduction to Linear Algebra*. Wellesley-Cambridge Press.

Some Advice

Guidance for Success

- Attend/Watch Lectures
- Engage with Tutorials
- Ask Questions
- Read Suggested Books
- Use Online Resources (Google, YouTube, etc...)
- Keep Your Work Organised
- Always Ask for Help

Topic One
Sets, Functions, and Writing Mathematically



Lecture Contents

By the end of this session you will be able to...

1. Learn about concepts around learning mathematics
2. Understand sets, functions, and number systems
3. Know how to read and write mathematics

The Aim

Learn concepts that will be

1. Needed as you go through your studies
2. Continually used here or other modules
3. Developed into more sophisticated concepts later

these include

1. Concepts around knowledge and learning
2. Basics constituents of mathematics: sets, functions, and number systems
3. Writing mathematics precisely and beautifully

Research

Spend some time to think/google about the truth or falsehood of the following statements.

1. Being good at a subject is a matter of inborn talent rather than hard work
2. Knowledge is composed of isolated facts
3. Divided attention, a result of multi-tasking, is effective for learning compared to focusing on one task at a time

On Inborn Talents and Isolated Facts

Talents

Believing ability is inborn reduces industriousness and perseverance

- You seek to avoid failure rather than **work for success**
- The more **constructive belief** is "I have to work especially hard at understanding this topic"
- Learning is not fast: **requires patience** and practice
- **Stay confident** that you'll make it alright through the path

Mathematical Knowledge

A **modern mathematician** has a good understanding of

- Pure/applied mathematics
- Statistics
- Programming

Exercise 1: Research

Think about the following items carefully and by conducting some research online or otherwise find which has a more significant effect in learning deeply.

1. Listen to lectures and work on problem sheets
2. Work out things by yourself
3. Learn, relearn, and memorise
4. Teaching others
5. Ask questions, discuss, and debate

Metacognition

Is "thinking about thinking" and "knowing about knowing"

- Good students know when they have mastered material
- Weaker students tend to be grossly overconfident

The entire mathematics is concerned with collections of objects, **sets**, and correspondences between them, **functions**. In this section we briefly introduced these.

Definition (Set)

A set is a well-defined collection of objects. The objects in the set are called the elements or members of the set.

Remark 1: Membership

- We usually define a particular set by making a list of its elements between brackets.
- If x is a member of the set X, then we write $x \in X$. We read this as 'x is an element (or member) of X' or 'x is in X'.
- If x is not a member, then we write $x \notin X$.

Example

1. The set containing the numbers $1, 2, 3, 4$ and 5 is written $\{1, 2, 3, 4, 5\}$. The number 3 is an element of the set, i.e., $3 \in \{1, 2, 3, 4, 5\}$, but $6 \notin \{1, 2, 3, 4, 5\}$. Note that we could have written the set as $\{3, 2, 5, 4, 1\}$ as the order of the elements is unimportant.
2. The set $\{\text{dog}, \text{cat}, \text{mouse}\}$ is a set with three elements: dog, cat and mouse.
3. The set $\{1, 5, 12, \{\text{dog}, \text{cat}\}, \{5, 72\}\}$ is the set containing the numbers $1, 5, 12$ and the sets $\{\text{dog}, \text{cat}\}$ and $\{5, 72\}$. Note that sets can contain sets as members.

Exercise 2: Sets and Membership

Consider the set $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$. How many elements does it have? Does X contain number 2? How about number 3?

More on Sets

Definition

- **Empty Set.** The set with no elements is called the empty set and is denoted $\emptyset$.
- **Equality.** Two sets are equal if they have the same elements. If set X equals set Y, then we write $X = Y$. If not we write $X \neq Y$.
- **Cardinality.** If the set X has a finite number of elements, then we say that X is a finite set. If X is finite, then the number of elements is called the cardinality of X and is denoted $|X|$.

Example

1. The sets $\{5, 7, 15\}$ and $\{7, 15, 5\}$ are equal, i.e., $\{5, 7, 15\} = \{7, 15, 5\}$. They have cardinality 3.
2. The sets $\{1, 2, 3\}$ and $\{2, 3\}$ are not equal, i.e., $\{1, 2, 3\} \neq \{2, 3\}$. Fist has cardinality 3 second has cardinality 2.
3. The sets $\{2, 3\}$ and $\{\{2\}, 3\}$ are not equal. They both have cardinality 2.
4. We have $\emptyset \neq \{\emptyset\}$, $|\emptyset| = 0$, and $|\{\emptyset\}| = 1$.

Exercise 3: Cardinality

Find the cardinality of the sets $\{\emptyset, 3, 4, \text{cat}\}$ and $\{\emptyset, 3, \{4, \text{cat}\}\}$. Are they equal as sets?

Subsets

Definition (Subset)

Suppose that X is a set. A set Y is a subset of X if every element of Y is an element of X. We write $Y \subseteq X$. This is the same as saying that, if $x \in Y$, then $x \in X$.

Example

1. The set $Y = \{1, \{3, 4\}, \text{mouse}\}$ is a subset of $X = \{1, 2, \text{dog}, \{3, 4\}, \text{mouse}\}$
2. The set $\{1, 2, 3\}$ is not a subset of $\{2, 3, 4\}$ or $\{2, 3\}$.
3. For any set X we have $\emptyset \subseteq X$ and $X \subseteq X$.

Remark 2: Subsets vs Membership

It is important that you distinguish between being an element of a set and being a subset of a set. If $x \in X$, then $\{x\} \subseteq X$.

Exercise 4: Subsets

Find all subsets of the set $\{1, \{3, 4\}, \text{mouse}\}$.

Proper Subsets and Defining Sets

Definition (Proper Subsets)

A subset Y of X is called a proper subset of X if Y is not equal to X. We denote this by $Y \subset X$.

Example

The set $\{1, 2, 5\}$ is a proper subset of $\{-6, 0, 1, 2, 3, 5\}$. For any set X, the subset X is not a proper subset of X.

Definition (Defining Sets)

We can define sets using a different notation of $\{x|x \text{ satisfies property } P\}$. The symbol '$|$' is read as 'such that'. Sometimes the colon ':' is used in place of '$|$'.

Example

Let $X = \{-6, 0, 1, 2, 3, 5\}$ and $Y = \{x|x \in X \text{ and } x < 2\}$. Then we have $Y = \{-6, 0, 1\}$.

Number Systems

There are several different sets of numbers. All of them have infinite cardinality. These are the following sets

- **Natural numbers** $\{1, 2, 3, 4, ...\}$ denoted by $\mathbb{N}$. The dots mean that we go on forever and can be read as 'and so on'.
- **Integers** $\{..., -4, -3, -2, 0, 1, 2, 3, 4, ...\}$ and is denoted by $\mathbb{Z}$.
- **Rational numbers** denoted by $\mathbb{Q}$ and consists of all fractional numbers, i.e., $x \in \mathbb{Q}$ if x can be written in the form $\dfrac{p}{q}$ where p and q are integers with $q \neq 0$.
- **Real numbers** denoted by $\mathbb{R}$ any number that can be given a decimal representation for example this include $\sqrt{2}$, π, e.
- **Complex numbers** denoted $\mathbb{C}$ can be written by the defining set $\{x + yi | x, y \in \mathbb{R},\ i = \sqrt{-1}\}$.

Functions

Definition (Function)

Let X and Y be sets. A **function** from X to Y is a rule that **assigns** to each $x \in X$ a single element of Y, denoted by $f(x)$. We write

$$f : X \longrightarrow Y$$

to mean that f is a function from X to Y. If $f(x) = y$, we often say f sends $x \mapsto y$.

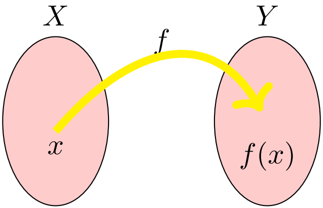
Example

The function

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^2 \end{aligned}$$

squares every real number. We sometimes write $f(x) = x^2$, so for example $f(1) = 1, f(\sqrt{2}) = 2, f(3) = 9$, etc...

The set X is known as the **domain** and Y the **range**.



The elements of Y which can be reached by applying f to elements of X for a set called the **image** of f. That is

$$\operatorname{Im} f = f(X) = \{f(x) \mid x \in X\}.$$

Exercise 4: Functions

Find domain, range, and image of the function

$$\begin{aligned} f : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ x &\longmapsto x^2. \end{aligned}$$

Lets begin first by making some remarks mathematical books and how to use them effectively. Keep the following points in mind:

- Read with a purpose
- Choose a book at the right level
- Read with pen and paper at hand
- Don't need to read it like a novel

in reading long pieces you may

- Skim through and identify what is important
- Ask questions and check other sources
- Read through carefully. You can do statements first and proofs later if you like
- Be active. This should include checking the text and doing the exercises
- Reflect and write summaries

General Rules

The primary rule is that you should write in simple, correctly punctuated sentences.

- Write in sentences and avoid symbols that may not be understood by readers.
- Use punctuation. The purpose of punctuation is to make the sentence clear.
- Keep it simple. To achieve this, use short words and sentences. Short sentences are easy to read.
- Explain what you are doing – keeping the reader informed
- Don't draw arrows everywhere. Write in lines and not in clusters which are joined by arrows.
- Proofread your work, reflect on each step, and make sure each step is clear.

Expressing Yourself Clearly

Examples of ways to improve the presentation of mathematics are given below

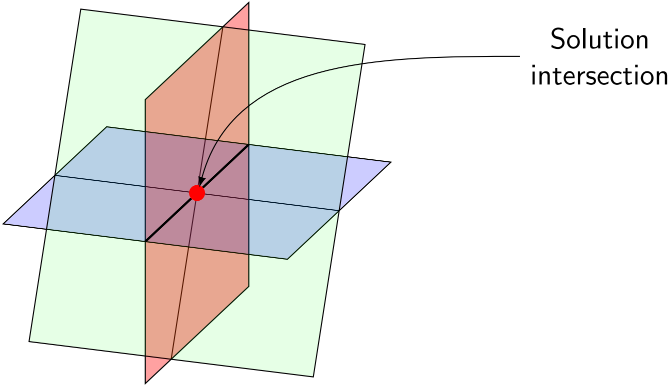
- If you use ‘if’, then use ‘then’.
- Not everything is a ‘formula’: Call things by their correct name.
- Avoid writing ‘it’ and try to refer to the object itself.
- Decimal approximations: avoid writing general long decimal representations $\sin 7 = 0.656986598$ you may write $\sin 7 \simeq 0.657$.
- Don’t begin sentences with a symbol, for example, avoid ‘f is a function with domain $\mathbb{R}$.’
- Do not use the implication symbol $\Longrightarrow$ or : instead of $=$.
- Use common symbols and notation and define your symbols carefully.

Summary: What we did today...

Basics Facts Knowledge	Inborn talents and isolated facts
Learning Deaply	Work out, discussions, teach others
Sets and Subsets	Membership, cardinality, equality
Number Systems	Natural, integers, rational, real, ...
Functions	Domain, range, image
Writing Mathematically	Rules
Next Topic	Linear Equations and Vectors I

Topic Two
Linear Equations and Vectors I[a]

[a](Lay et al., 2015, Sections 1.1, 1.2)



Lecture Contents

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn the basics and applications of linear algebra
2. Define systems of linear equations and their consistency
3. Find the augmented matrix for a system of linear equations
4. Preform elementary row operation on augmented matrices
5. Find echelon form matrices by hand or computer packages

Group Activity

- Access moodle
- Find your group
- Find the book
- With you group write a two paragraphs summary on the introductory example on the chapter depending on your group spend 10 min
- Present it for 5 min
- Upload your paragraph to the discussion section of moodle

Linear Equations

A linear equation in the variables $x_1, ..., x_n$ is an equation that can be written in the form

$$a_1x_1 + \cdots + a_nx_n = b$$

where b and the **coefficients** $a_1, ..., a_n, b$ can be real or complex numbers. The subscript n may be any positive integer.

Example

The equations

$$x_1 + \sqrt{3} = 2x_2 \text{ and } 2x_1 + 3x_2 + 10x_3 = 0$$

are linear whereas

$$x_1x_3 + \sqrt{3} = 2x_2 \text{ and } 2x_1 + 3x_2^2 + 10x_3 = 0$$

are not.

- A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables—say, $x_1, ..., x_n$
- An example is

$$\begin{aligned} 2x_1 + 3x_2 + x_3 &= 1 \\ 5x_2 + 3x_4 &= 0 \end{aligned}$$

- A **solution** of the system is a list $s_1, ..., s_n$ of numbers that makes each equation a true statement when the values $s_1, ..., s_n$ are substituted for $s_1, ..., s_n$, respectively
- The set of all possible solutions is called the **solution set** of the linear system
- Two linear systems are called **equivalent** if they have the same solution set

Consistency

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Example

- The equations $x_1 = x_2$ and $x_1 = x_2 + 2$ has no solution
- The equations $x_1 = -x_2$ and $x_1 = x_2 + 2$ has exactly one solution
- The equations $x_1 = -x_2$ and $2x_1 = -2x_2$ has infinitely many solutions

Matrix Notation

- The essential information of a linear system can be recorded compactly in a rectangular array called a matrix.
- Given a system of equations

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 8x_2 &= -1 \\ -4x_2 + 6x_3 &= 7 \end{aligned}$$

- The matrix $\begin{bmatrix} 5 & 2 & 1 \\ 1 & -8 & 0 \\ 0 & -4 & 6 \end{bmatrix}$ is called the **coefficient matrix**
- And the matrix $\begin{bmatrix} 5 & 2 & 1 & 0 \\ 1 & -8 & 0 & -1 \\ 0 & -4 & 6 & 7 \end{bmatrix}$ is called the **augmented matrix**.

Exercise 1: Natural Sounding Applications

Alice and Bob buy fruits. Alice buys two apples and six oranges for \$4. Bob buys three apple and five oranges for \$5. How much does each apple and orange cost?

Exercise 2: Solving a Linear System

Solve the system below.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 8x_2 &= 8 \\ 5x_1 - 5x_3 &= 10 \end{aligned}$$

The following operations are valid for solving systems of linear equations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row
2. (Interchange) Interchange two rows
3. (Scaling) Multiply all entries in a row by a nonzero constant

Existence and Uniqueness Questions

1. Is the system consistent; that is, does at least one solution exist?
2. If a solution exists, is it the only one; that is, is the solution unique?

Determine if the following system is consistent

$$\begin{aligned} 2x_2 + 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 4x_1 - 8x_2 + 12x_3 &= 1 \end{aligned}$$

A rectangular matrix is in **echelon** form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it
3. All entries in a column below a leading entry are zeros

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon** form (or reduced row echelon form):

1. The leading entry in each nonzero row is 1
2. Each leading 1 is the only nonzero entry in its column

Pivot

Definition

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Column 1 is the pivot column in the matrix below

$$\begin{bmatrix} 5 & 2 & 1 & 0 \\ 1 & -8 & 0 & -1 \\ 0 & -4 & 6 & 7 \end{bmatrix}$$

The Row Reduction Algorithm

1. Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.
2. Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.
3. Use row replacement operations to create zeros in all positions below the pivot.
4. Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.
5. Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

Existence and Uniqueness

- A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column—that is, if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & b \end{bmatrix} \text{ with } b \text{ nonzero}$$

- If a linear system is consistent, then the solution set contains either
  1. a unique solution, when there are no free variables, or
  2. infinitely many solutions, when there is at least one free variable.

Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

You may use **Colab** for Python the relevant library is SumPy.

```
import sympy
M=sympy.Matrix(
[[5, 2, 1, 0],
[1, -8, 0, -1],
[0, -4, 6, 7]])
M.rref()

>>> M.rref()
(Matrix([
[1, 0, 0,   -9/32],
[0, 1, 0,  23/256],
[0, 0, 1, 157/128]]), (0, 1, 2))
```

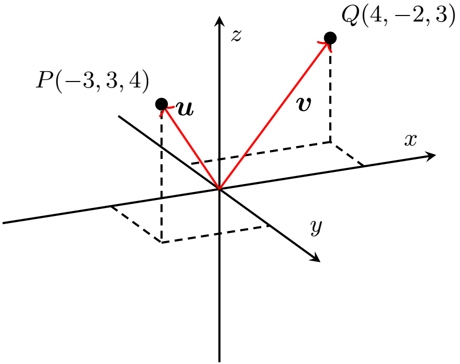
You may use **RStudio Cloud** for R. The relevant libraries is matlib.

```
library(matlib)
A <- matrix(
c(5, 2, 1,
1, -8, 0,
0, -4, 6), 3, 3, byrow=TRUE)
b <- c(0, -1, 7)
echelon(A, b, verbose=TRUE, fractions=TRUE)
...
multiply row 3 by 1/42 and subtract from row 2
        [,1]    [,2]    [,3]    [,4]
[1,]       1       0       0   -9/32
[2,]       0       1       0  23/256
[3,]       0       0       1 157/128
```

- Linear Algebra
  - Introduction, applications
- Systems of Linear Equations
  - Consistency, matrix notation
- Elementary Row Operations
  - Echelon form, R and Python
- Next Time
  - Linear Equations and Vectors II

Topic Three
Linear Equations and Vectors II[a]

[a] (Lay et al., 2015, Sections 1.3-1.6)



Lecture Contents

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn about vector and operation on them
2. Define span, linear combinations, and solve vector equations
3. Find solution sets of linear systems by hand or computer package
4. Use linear algebra to study problems in economy, chemistry and network flow

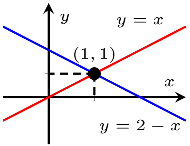
Question

What is linear algebra?

- Linear algebra arises from a need to solve **systems of linear equations**

$$x + y = 2$$
$$x - y = 0$$

Algebraically

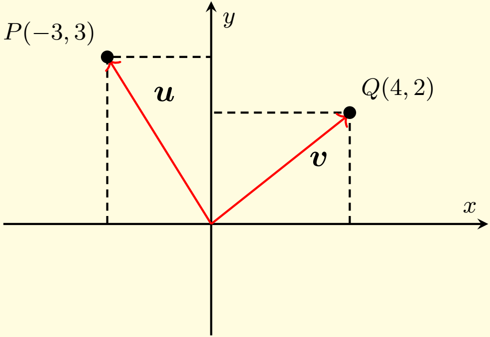


Geometrically

- Linear algebra plays an important role in many areas of pure and applied mathematics
- Computers these days solve systems with thousands of linear equations every minute

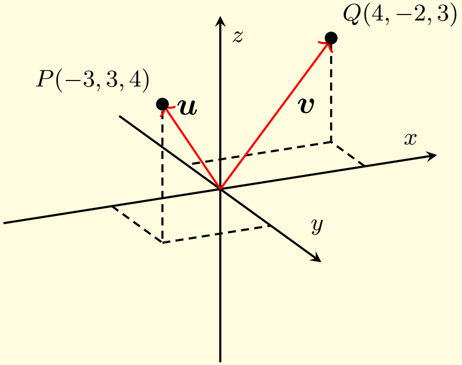
What is a vector?

Think about the 2-dimensional space $\mathbb{R}^2$



What is a vector?

Think about the 3-dimensional space $\mathbb{R}^3$



Geometry: Intuition

- Many physical quantities, such as temperature and speed, possess only **magnitude**
- These quantities can be represented by **real numbers** and are called **scalars**
- Vectors have **magnitude** and **direction**
- They are represented by tuples, for example,

$$\boldsymbol{u} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix}, \ \boldsymbol{v} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$$

- We may use bold letter $\boldsymbol{u}$ or letters with underline $\underline{u}$ to denote a vector

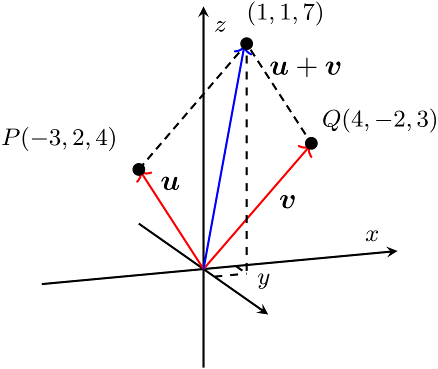
Algebraically the result of adding two vectors is **component-wise addition**. For example,

if

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

then

$$\boldsymbol{a} + \boldsymbol{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}.$$



Geometrically the result of adding two vectors is obtained by the **parallelogram law**.

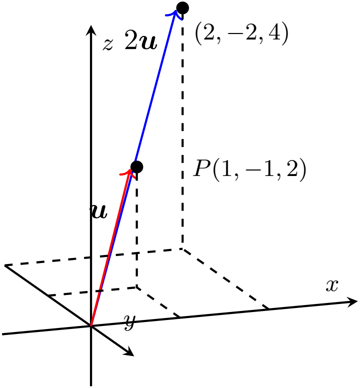
Algebraically the result of multiplying a vector by a scalar λ is **component-wise**. For example,

if

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix},$$

then

$$\lambda\boldsymbol{a} = \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \\ \lambda a_3 \end{bmatrix}.$$



Geometrically the result of adding two vectors is obtained by **scaling the vector**, changing direction if $\lambda < 0$.

The n-dimensional Euclidean Space

For a natural number n let $\mathcal{V} = \mathbb{R}^n$ with addition and scalar multiplication

$$(u_1, u_2, ..., u_n) + (v_1, v_2, ..., v_n) = (u_1 + v_1, u_2 + v_2, ..., u_n + v_n)$$
$$\lambda(u_1, u_2, ..., u_n) = (\lambda u_1, \lambda u_2, ..., \lambda u_n),$$
$$\mathbf{0} = (0, 0, ..., 0).$$

In such case for $u = (u_1, u_2, ..., u_n)$ the vector $\widetilde{u}$ such that $u + \widetilde{u} = \mathbf{0}$ is give by

$$-u = (-u_1, -u_2, ..., -u_n).$$

Exercise 1: Vector Operations

Let $\boldsymbol{u} = (2, 4, -5, 1)$ and $\boldsymbol{v} = (1, 2, 3, 4)$. Find

$$\boldsymbol{u} + \boldsymbol{v},\ 3\boldsymbol{v},\ -\boldsymbol{v},\ 2\boldsymbol{u} - 3\boldsymbol{v}.$$

Properties of Vectors in $\mathbb{R}^n$

Vectors is $\mathbb{R}^n$ form a set $\mathcal{V}$, with elements $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \ldots$, together with addition $+$ and a scalar multiplication so that

$$\boldsymbol{u} + \boldsymbol{v} \in \mathcal{V} \text{ and } \lambda\boldsymbol{u} \in \mathcal{V} \text{ for all } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V},\ \lambda \in \mathbb{R}.$$

In addition, for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$ and $\lambda, \mu \in \mathbb{R}$ the following **axioms** are satisfied.

Group Axioms

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
3. There exists $\mathbf{0} \in \mathcal{V}$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
4. There exists $\widetilde{\boldsymbol{u}} \in \mathcal{V}$ such that $\boldsymbol{u} + \widetilde{\boldsymbol{u}} = \mathbf{0}$
   $\widetilde{\boldsymbol{u}}$ is denoted by $-\boldsymbol{u}$

Scalar Axioms

5. $\lambda(\boldsymbol{u} + \boldsymbol{v}) = \lambda\boldsymbol{u} + \lambda\boldsymbol{v}$
6. $(\lambda + \mu)\boldsymbol{u} = \lambda\boldsymbol{u} + \mu\boldsymbol{u}$
7. $\lambda(\mu\boldsymbol{u}) = (\lambda\mu)\boldsymbol{u}$
8. $1\boldsymbol{u} = \boldsymbol{u}$

Exercises

Exercise 2: Vector Equality

Two vectors $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ and $\boldsymbol{v} = (v_1, v_2, ..., v_n)$ are equal if $u_i = v_i$ for every $i = 1, ..., n$. Find x, y, z so that

$$(x - y, x + z, z - 1) = (1, 2, 3).$$

Exercise 3: Function Spaces

Let X be a set and $\mathrm{M}(X, \mathbb{R})$ the set of all functions $f : X \longrightarrow \mathbb{R}$ with addition and scalar multiplication

$$(f + g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda f(x).$$

Show $\mathrm{M}(X, \mathbb{R})$ satisfies the 8 axioms on the previous slide.

Given vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_p \in \mathbb{R}^n$ and scalars $c_1, ..., c_p \in \mathbb{R}$ the vector defined by

$$\boldsymbol{y} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

is called a linear combination of $\boldsymbol{v}_1, ..., \boldsymbol{v}_p$ with weights $c_1, ..., c_p$.

Example

1. Write $\boldsymbol{b} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$ as a linear combination of $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
2. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$. Determine if $\boldsymbol{y} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$ can be written as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

Note that a vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \boldsymbol{b} \end{bmatrix}$$

Definition

If $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \in \mathbb{R}^n$, then the set of all linear combinations of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$ denoted by $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p\}$ is called the subset of $\mathbb{R}^n$ generated or spanned by $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$, that is the collection of all vectors that can be written in the form

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

for some scalars $c_1, c_2, ..., c_p \in \mathbb{R}$.

Example

Find the subset of $\mathbb{R}^3$ generated by

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \ \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Definition

If A is an $m \times n$ matrix, with columns $\boldsymbol{a}_1, ..., \boldsymbol{a}_n$, and if $\boldsymbol{x}$ is in $\mathbb{R}^n$, then the product of A and $\boldsymbol{x}$, denoted by $A\boldsymbol{x}$, is the linear combination of the columns of A using the corresponding entries in $\boldsymbol{x}$ as weights; that is,

$$A\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n$$

Example

Find the following

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Theorem

If A is an $m \times n$ matrix, $\boldsymbol{u}$ and $\boldsymbol{v}$ are either vectors in R^n and c is a scalar, then we have the following.

1. $A(\boldsymbol{u} + \boldsymbol{v}) = A\boldsymbol{u} + A\boldsymbol{v}$, *here matrix addition is component-wise*
2. $A(\lambda \boldsymbol{u}) = \lambda A\boldsymbol{u}$

Theorem

If A is an $m \times n$ matrix with columns $\boldsymbol{a}_1, ..., \boldsymbol{a}_n$, and if $\boldsymbol{b} \in \mathbb{R}^m$, the matrix equation

$$A\boldsymbol{x} = \boldsymbol{b}$$

has the same solutions as the vector equation

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \boldsymbol{b} \end{bmatrix}$$

Note the above equation has a solution if and only if $\boldsymbol{b}$ can be written as a linear combination of columns of A, i.e., if

$$\boldsymbol{b} \in \text{Span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_n\}.$$

Given A a matrix as above. Does the equation $A\boldsymbol{x} = \boldsymbol{b}$ always have a solution?

Theorem

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

1. *For each $b \in \mathbb{R}^m$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution*
2. *Each $\boldsymbol{b} \in \mathbb{R}^m$ is a linear combination of the columns of A*
3. *The columns of A span $\mathbb{R}^m$*
4. *A has a pivot position in every row*

Solution Sets of Linear Systems

- A system of linear equations is said to be **homogeneous** if it can be written in the form $A\boldsymbol{x} = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in $\mathbb{R}^m$.
- A homogeneous system always a solution, zero solution usually called the trivial solution.
- Has a nontrivial solution if and only if the equation has at least one free variable

Example

Determine if the following homogeneous system has a nontrivial solution. Then describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Theorem

Suppose the equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for some given $\boldsymbol{b}$, and let $\boldsymbol{p}$ be a solution. Then the solution set of $A\boldsymbol{x} = \boldsymbol{b}$ is the set of all vectors of the form $\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{v}_h$, where $\boldsymbol{v}_h$ is any solution of the homogeneous equation $A\boldsymbol{x} = 0$.

Applications

1. Equilibrium Prices in Economics
2. Balancing Chemical Equations
3. Network Flow

Equilibrium Prices in Economics

Suppose an economy consists of the Coal, Electric (power), and Steel sectors, and the output of each sector is distributed among the various sectors and the total output of the

- Electric sector goes 40% to Coal, 50% to Steel, and the remaining 10% to Electric
- Steal sector goes 60% to Coal, 20% to Steel, and the remaining 20% to Electric
- Coal sector goes 60% to Electric, 40% to Steel

There exist **equilibrium prices** that can be assigned to the total outputs of the various sectors in such a way that the income of each sector exactly balances its expenses.

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C3H8) combines with oxygen (O2) to form carbon dioxide (CO2) and water (H2O), according to an equation of the form

$$x_1 C_3 H_8 + x_2 O_2 \longrightarrow x_3 CO_2 + x_4 H_2 O$$

To "balance" this equation, a chemist must find whole numbers $x_1, ..., x_4$ such that the total numbers of carbon (C), hydrogen (H), and oxygen (O) atoms on the left match the corresponding numbers of atoms on the right (because atoms are neither destroyed nor created in the reaction).

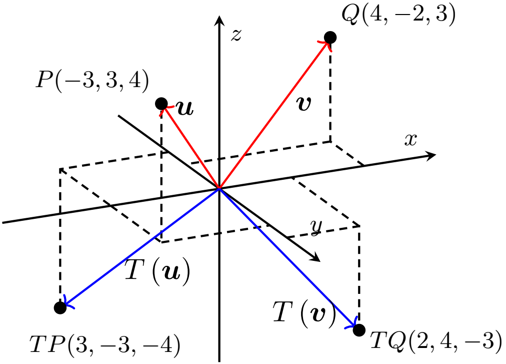
Summary: What we did today...

- Linear algebra
  - History and applications
- Basics of vectors in $\mathbb{R}^n$
  - Addition and scalar multiplication
- Vector axioms
  - Euqality of vectors
- Linear Combinations
  - Vector equations, span
- Matrix Equation $A\boldsymbol{x} = \boldsymbol{b}$
  - Solution sets, applications
- Next time
  - Matrix Algebra I

Topic Four
Matrix Algebra I[a]

[a](Lay et al., 2015, Sections 1.7-1.9)

$$T\boldsymbol{x} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \boldsymbol{x}$$



Lecture Contents

By the end of this session you will be able to...

1. Learn about linear independence
2. Understand linear transformation

Linear Dependence

- Last time we talked about the span of a set of vectors that is for if $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \in \mathbb{R}^n$

  $$\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p\} = \{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p : c_1, ..., c_p \in \mathbb{R}\}$$

- Note that we can always write 0 vector as the linear combination of vectors above by setting $c_1 = c_2 = \cdots = c_p = 0$
- However, if $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$ are linearly dependent, we can find $c_1, c_2, ..., c_p$ not all equal to zero such that

  $$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = 0$$

- For example,

  $$\begin{bmatrix} 2 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

Linear Independence Motivation

- Now if $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$ say we have $c_1, c_2, ..., c_p$ with $c_1 \neq 0$ and

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = 0$$

- Then we can write

$$-\frac{c_2}{c_1}\boldsymbol{v}_2 - \cdots - \frac{c_p}{c_1}\boldsymbol{v}_p = \boldsymbol{v}_1$$

- Which means $\text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p\} = \text{Span}\{\boldsymbol{v}_2, ..., \boldsymbol{v}_p\}$
- On the other hand, when $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p$ are linearly independent the above procedure is not possible!

Linear Independence Definition

Definition

An indexed set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \in \mathbb{R}^n$ is said to be linearly independent if the vector equation

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p = 0$$

has only the trivial solution i.e., $c_1 = c_2 = \cdots = c_p = 0$. Otherwise the set is said to be linearly dependent.

Example

Show that the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent.

Exercise

1. Determine if the set below is linear independent.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

2. Determine if the columns of the matrices below are linearly independent.

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 & 1 \\ 1 & 2 & -1 & 3 \\ 5 & 8 & 0 & 4 \end{bmatrix}$$

You can show in general that if a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \in \mathbb{R}^n$ is linearly dependent if $p > n$.

Example

Suppose a set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ are linearly independent. Then the vectors $\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 + \boldsymbol{v}_3 \in \mathbb{R}^3$ are linearly independent.

Example

Suppose a set of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \in \mathbb{R}^3$ span $\mathbb{R}^3$. Then the vectors $\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 + \boldsymbol{v}_3 \in \mathbb{R}^3$ also span $\mathbb{R}^3$.

- Linear transformations are certain functions on a given subset of vectors
- For example, given an $m \times n$ matrix with m rows and n columns for each vector $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$\boldsymbol{b} = A\boldsymbol{x} \in \mathbb{R}^m$$

- This operation is providing a function

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \boldsymbol{x} &\longmapsto A\boldsymbol{x} \end{aligned}$$

  known as **linear a transformation**
- Note, not all transformations are linear for example $(x_1, x_2) \longmapsto (x_1^2 + x_2, x_2)$
- Later we will see that the linear transformations are those given by matrices

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ define a transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then

1. Find $T(\boldsymbol{u})$, the image of u under the transformation T
2. Find an $\boldsymbol{x} \in \mathbb{R}^2$ whose image under T is $\boldsymbol{b}$
3. Is there more than one $\boldsymbol{x}$ whose image under T is $\boldsymbol{b}$?
4. Determine if $\boldsymbol{c}$ is in the range of the transformation T

Properties of Linear Transformation

A transformation (or mapping) T is linear if

1. $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v}$ in the domain of T
2. $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$ for all $\boldsymbol{u}$ in the domain of T and any scalar c

It follows from the two properties above that

$$\begin{aligned} T(0) &= 0 \\ T(c\boldsymbol{u} + d\boldsymbol{v}) &= cT(\boldsymbol{u}) + dT(\boldsymbol{v}) \\ T(c_1\boldsymbol{v}_1 + \cdots + c_p\boldsymbol{v}_p) &= c_1T(\boldsymbol{v}_1) + \cdots + c_pT(\boldsymbol{v}_p) \end{aligned}$$

The Matrix of a Linear Transformation

- Whenever a linear transformation T arises geometrically or is described in words, we usually want a "formula" for $T(\boldsymbol{x})$
- every linear transformation from $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is actually a matrix transformation $\boldsymbol{x} \longmapsto A\boldsymbol{x}$
- Then T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that $T(\boldsymbol{x}) = A\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$. In fact, A is the $m \times n$ matrix whose j the column is the vector $T(\boldsymbol{e}_j)$, where $\boldsymbol{e}_j$ is the j th column of the identity matrix in $\mathbb{R}^n$,

$$A = \begin{bmatrix} T(\boldsymbol{e}_1) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}.$$

Examples

1. Find the standard matrix A for the dilation transformation $T(\boldsymbol{x}) = 3\boldsymbol{x}$, for $\boldsymbol{x} \in \mathbb{R}^3$.
2. Find a matrix for a linear transformation which maps

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} a \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ b \end{bmatrix}$$

is there a unique linear transformation for the above mapping?

Geometric Linear Transformations

You can interpret linear transformation geometrically. Some examples of linear transformation in $\mathbb{R}^2$ are as follows

- Reflection through the x-axis $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- Reflection through the y-axis $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
- Reflection through the $y = x$ line $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Horizontal h and vertical k contraction/expansion $\begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}$
- Horizontal h and vertical k sheer $\begin{bmatrix} 1 & h \\ k & 1 \end{bmatrix}$
- Horizontal and vertical projection $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Onto and One-to-One Linear Transformation

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\boldsymbol{b} \in \mathbb{R}^m$ is the image of at least one $\boldsymbol{x} \in \mathbb{R}^n$, i.e., $T(\boldsymbol{x}) = \boldsymbol{b}$.

Example

Consider the linear transformations

$$\begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Definition

A mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be one-to-one if each $\boldsymbol{b} \in \mathbb{R}^m$ is the image of at most one $\boldsymbol{x} \in \mathbb{R}^n$, i.e., $T(\boldsymbol{x}) = \boldsymbol{b}$ has a unique solution.

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then T is one-to-one if and only if the equation $T(\boldsymbol{x}) = 0$ has only the trivial solution.

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Let A be the matrix of T. Then

1. *T is onto if and only if columns of A space $\mathbb{R}^m$*
2. *T is one-to-one if and only if columns of A are linearly independent*

Example

Let $T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, 2x_1 + 3x_2)$. Show that T is a one-to-one linear transformation. Does T map $\mathbb{R}^2$ onto $\mathbb{R}^3$?

Summary: What we did today...

- Linear Independence
  - Examples
- Linear Transformation
  - Matrix, one-to-one, onto
- Next time
  - Matrix Algebra II

Topic Five
Matrix Algebra II[a]

[a] (Lay et al., 2015, Sections 2.1-2.3)

$$
\begin{aligned}
A &= \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix},\ \boldsymbol{a}_1, ..., \boldsymbol{a}_n \in \mathbb{R}^m \\
B &= \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_k \end{bmatrix},\ \boldsymbol{b}_1, ..., \boldsymbol{b}_k \in \mathbb{R}^n \\
C &= \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \end{bmatrix},\ \boldsymbol{c}_1, ..., \boldsymbol{c}_n \in \mathbb{R}^m \\
AB &= \begin{bmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & \cdots & A\boldsymbol{b}_k \end{bmatrix} \\
A + C &= \begin{bmatrix} \boldsymbol{a}_1 + \boldsymbol{c}_1 & \boldsymbol{a}_2 + \boldsymbol{c}_2 & \cdots & \boldsymbol{a}_n + \boldsymbol{c}_n \end{bmatrix}
\end{aligned}
$$

Lecture Contents

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn about operations on matrices: addition and multiplication
2. Compute transpose and inverse of a matrix
3. Use computer packages to compute matrix operations

- In the past few weeks we have worked with matrices in solving systems of linear equations or as linear transformation
- We usually denote an $m \times n$ matrix A using the column vectors

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}, \ \boldsymbol{a}_1, ..., \boldsymbol{a}_n \in \mathbb{R}^m$$

- Note sometimes the scalar entries in the ith row and jth column is denoted by a_{ij}
- This week we shall learn how to preform several important operations on matrices including addition and matrix multiplication

Sums and Scalar Multiples

- Two matrices can can be added if they have exactly the same number of rows and columns
- If A and B are $m \times n$ matrices, then the sum $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B
- If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A

Example

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 & 0 & 4 \\ 5 & 0 & -1 & 8 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 0 & -1 \\ 4 & 8 & 2 \end{bmatrix}.$$

Calculate $A + B, \; A + C, \; -A, \; 2C$ if possible.

Let A, B, and C be matrices of the same size, and let r and s be scalars. The we have the following

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + 0 = A$
4. $r\,(A + B) = rA + rB$
5. $(r + s)\,A = rA + sA$
6. $r\,(sA) = rsA$

- Recall a matrix B multiplies a vector $\boldsymbol{x}$, it transforms $\boldsymbol{x}$ into the vector $B\boldsymbol{x}$
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A\,(B\boldsymbol{x})$
- We want to represent this composite mapping as multiplication by a single matrix, denoted by AB
- Suppose A is $m \times n$, B is $n \times p$, and $\boldsymbol{x} \in \mathbb{R}^p$
- Denote the columns of B by $\begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_p \end{bmatrix}$ and the entries in $\boldsymbol{x}$ by $x_1, ..., x_p$

- Note from the definition of $B\boldsymbol{x}$ we had that

$$B\boldsymbol{x} = x_1\boldsymbol{b}_1 + \cdots + x_p\boldsymbol{b}_p$$

- Now using properties the product of a matrix with a vector for $A\left(B\boldsymbol{x}\right)$ we have

$$A\left(B\boldsymbol{x}\right) = Ax_1\boldsymbol{b}_1 + \cdots + Ax_p\boldsymbol{b}_p = x_1A\boldsymbol{b}_1 + \cdots + x_pA\boldsymbol{b}_p$$

- Therefore, multiplying $\boldsymbol{x}$ by $\begin{bmatrix} A\boldsymbol{b}_1 & \cdots & A\boldsymbol{b}_p \end{bmatrix}$ transforms $\boldsymbol{x}$ into $A\left(B\boldsymbol{x}\right)$

Definition

If A is $m \times n$ and $B = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_p \end{bmatrix}$ is $n \times p$, then the product AB is the $m \times p$ matrix whose columns are $A\boldsymbol{b}_1, ..., A\boldsymbol{b}_p$. That is,

$$AB = \begin{bmatrix} A\boldsymbol{b}_1 & \cdots & A\boldsymbol{b}_p \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 7 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Calculate AB, AC, CA, BA, BC, C^2 if possible.

Properties of Matrix Multiplication

Row Column Rule for Computing AB

If a matrix an $m \times n$ matrix A has entries a_{ij} and $n \times p$ matrix B has entries b_{kl}, then

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + \cdots + a_{in} b_{nj}$$

Properties

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B+C) = AB + AC$ (left distributive law)
3. $(B+C)A = BA + CA$ (right distributive law)
4. $r(AB) = (rA)B = A(rB)$ for any scalar r
5. $I_m A = A = A I_n$ (identity for matrix multiplication)

Warnings

- In general, $AB \neq BA$. Try

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$. Try

$$\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$$

- If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$. Try

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

- If A is an $n \times n$ matrix, then A^k denotes the product of k copies of A
- If A is nonzero and if $\boldsymbol{x} \in \mathbb{R}^n$; then $A^k\boldsymbol{x}$ is the result of left-multiplying $\boldsymbol{x}$ by A repeatedly k times
- Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T , whose columns are formed from the corresponding rows of A

Example

Let $A = \begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 0 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Calculate $A^2, B^3, B^T, C^T, CC^T, C^TC$.

Theorem

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

1. $\left(A^T\right)^T = A$
2. $(A + B) = A^T + B^T$
3. *For any scalar r,* $(rA)^T = rA^T$
4. $(AB)^T = B^T A^T$

Definition

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that $CA = I$ and $AC = I$ where $I = I_n$; the $n \times n$ identity matrix. In this case, C is an inverse of A, which is unique and is denoted by A^{-1}.

Example

Consider the two matrix $\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} -2 & -1 \\ -1 & 0 \end{bmatrix}$. Does $\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ have an inverse?

An Algorithm for Finding A^{-1} for 2×2

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc = 0$, *then the inverse of* A *is given by*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The quantity $ad - bc$ *is called the determinant of* A.

Example

Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem

If A is an invertible $n \times n$ matrix, then for each $\boldsymbol{b} \in \mathbb{R}^n$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, solve the equation $A\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Properties of Inverse

Theorem

1. *If A is an invertible matrix, then A^{-1} is invertible and* $\left(A^{-1}\right)^{-1} = A$
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,* $(AB)^{-1} = B^{-1}A^{-1}$
3. *If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,* $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

An Algorithm for Finding A^{-1}

Algorithm

Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A can be reduced by elementary row operations to I, then $\begin{bmatrix} A & I \end{bmatrix}$ can be reduced by elementary row operations to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Example

Find the inverse of the matrix $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 3 \\ 4 & -6 & 0 \end{bmatrix}$ if it exists.

- Recall that matrix multiplication corresponds to composition of linear transformations
- When a matrix A is invertible, the equation $A^{-1}A\boldsymbol{x} = \boldsymbol{x}$ can be viewed as a statement about linear transformations
- A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be invertible if there exists a function $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$T\left(S\left(\boldsymbol{x}\right)\right) = \boldsymbol{x} = S\left(T\left(\boldsymbol{x}\right)\right)$$

Theorem

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S\left(\boldsymbol{x}\right) = A^{-1}\boldsymbol{x}$ is the unique function satisfying equations above equation for inverse.

You may use **Colab** for Python the relevant library is SumPy.

```
import sympy
a,b,c,d=sympy.symbols("a,b,c,d")
M=sympy.Matrix(
[[a, b],
[c, d]])
N=sympy.Matrix(
[[5, 2, 1, 0],
[1, -8, 0, -1]])
M*N # Multiplication
M.T # Transpose
M**-1 # Inverse
M.det() # Determinant
```

NumPy with Python

You may use **Colab** for Python the relevant library is NumPy.

```python
import numpy
M=numpy.array(
[[5, 2],
[1, -8]])
N=numpy.array(
[[5, 2, 1, 0],
[1, -8, 0, -1]])
numpy.matmul(M,N) # Multiplication
M.transpose() # Transpose
numpy.linalg.inv(M) # Inverse
numpy.linalg.det(M) # Determinant
```

You may use **RStudio Cloud** for R. The relevant libraries is matlib, but basic R functions work well in this case too.

```
M <- matrix(c(5, 2,
              1, -8), 2, 2, byrow=TRUE)
N <- matrix(c(5, 2, 1, 0,
              1, -8, 0, -1), 2, 4, byrow=TRUE)
M%*%N # Multiplication
t(M) # Transpose
solve(M) # Inverse
det(M) # Determinant
```

Summary: What we did today...

- Matrix Operations
  - Addition, scalar multiplication
- Matrix Multiplication
  - Properties and warnings
- Powers and Transpose
  - Inverse of a matrix, algorithm
- R and Python
  - SymPy, NumPy, matlib
- Next time
  - Determinants

Topic Six
Determinant[a]

[a] (Lay et al., 2015, Sections 3.1-3.3)

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix} = \begin{bmatrix} a_{ij} \end{bmatrix}, \ \boldsymbol{a}_1, ..., \boldsymbol{a}_n \in \mathbb{R}^n$$

$$\det A = \sum_j (-1)^{i+j} a_{ij} A_{ij}$$

$$A_{lm} = \begin{bmatrix} a_{ij} \end{bmatrix} \text{ for } i \neq l, j \neq m$$

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn how to calculate the determinant of a matrix
2. Understand the properties of determinants and calculate cofactor matrix
3. Learn the applications of determinant in matrix inversion
4. Develop geometric intuition for determinants of linear transformations

- We have been learning how to use matrices in order to solve systems of linear equation
- Also to use them as linear transformation of vectors
- Last week we looked at operation involving matrices including addition, multiplication, transpose, and inverse
- This week we shall learn an important invariant of a matrix known as the determinant in details
- We already mentioned that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is given by $\det A = ad - bc$
- Recall that A is invertible if and only if $\det A \neq 0$

Determinant

- Determinant can be calculated for any $n \times n$ matrix
- For example let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- The the determinant of A is given by

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Example

Compute the determinant of the following matrices

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 4 \\ 3 & 5 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Determinant in General

Definition

If $A = \left[a_{ij}\right]$ is an $n \times n$ matrix and if A_{ij} is formed from A by deleting the ith row and jth column, then the determinant of A is given by

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

Example

Calculate the determinant of

$$A = \begin{bmatrix} 2 & 1 & 4 & 5 \\ 3 & 5 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 4 & 1 & 0 & 0 \end{bmatrix}$$

Cofactor Matrix and Determinant

- The are alternative ways of computing the determinant using the cofactor matrices
- Suppose $A = \left[a_{ij}\right]$ is an $n \times n$ matrix and if A_{ij} is formed from A by deleting the ith row and jth column
- Then the (i, j)-cofactor of A is given by

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

- It turns out that the determinant of A can then be given by

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

  for any $i = 1, ..., n$
- Similarly

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + C_{nj}$$

  for any $j = 1, ..., n$

Calculate the determinant of

$$A = \begin{bmatrix} 2 & 1 & 4 & 5 & 4 \\ 3 & 5 & 0 & 0 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

Note using the cofactor definition it can be shown that if A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Interesting Note

- By today's standards, a 25×25 matrix is small
- Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion
- In general, a cofactor expansion requires more than $n!$ multiplications, and $25!$ is approximately 1.5×10^{25}
- If a computer performs one trillion multiplications per second, it would have to run for more than $500,000$ years to compute a 25×25 determinant by this method
- Fortunately, there are faster methods, as we'll soon discover

Properties of Determinant

Let A be a square matrix.

- If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$
- If two rows of A are interchanged to produce B, then $\det B = -\det A$
- If one row of A is multiplied by k to produce B, then $\det B = k \det A$
- To find $\det A$ you may reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries
- Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges
- If there are r interchanges, then we have

$$\det A = (-1)^r \det U$$

Theorem

Let A, B be $n \times n$ matrices.

1. *Then A is invertible if and only if* $\det A \neq 0$
2. *We have* $\det A^T = \det A$ *and* $\det A^{-1} = (\det A)^{-1}$
3. *We have* $\det AB = \det A \det B$

Example

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ verify

$\det A^T = \det A$, $\det A^{-1} = (\det A)^{-1}$, $\det AB = \det A \det B$.

Cramer's Rule

- Cramer's rule is needed in a variety of theoretical calculations
- For instance, it can be used to study how the solution of $A\boldsymbol{x} = \boldsymbol{b}$ is affected by changes in the entries of $\boldsymbol{b}$
- For any $n \times n$ matrix A and any $\boldsymbol{b} \in \mathbb{R}^n$, let $A_i(\boldsymbol{b})$ be the matrix obtained from A by replacing the ith column with $\boldsymbol{b}$

Theorem

Let A be an invertible $n \times n$ matrix. For any $\boldsymbol{b} \in \mathbb{R}^n$, the unique solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \boldsymbol{b}$ has entries given by

$$x_i = \frac{\det A_i(\boldsymbol{b})}{\det A}, \; i = 1, ..., n.$$

Example

Solve the system $2x_1 + x_2 = 6$ and $x_1 + 3x_2 = 7$ using Cramer's rule.

Applications of Cramer's Rule

- Cramer's rule can be used to find A^{-1} for an $n \times n$ matrix A
- Note if $\boldsymbol{x}$ is the jth column of A^{-1} we must have

$$A\boldsymbol{x} = \boldsymbol{e}_j$$

- Using Cramer's Rule we find

$$x_i = \frac{\det A_i(\boldsymbol{e}_j)}{\det A}$$

- Now properties of determinant imply that

$$\det A_i(\boldsymbol{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji}$$

- Therefore, we have

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} C_{ji}$$

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 0 & 1 \end{bmatrix}$$

Determinants can be interpreted geometrically.

Theorem

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Example

Calculate the area of the parallelogram determined by the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.

Determinants can be used to describe an important geometric property of linear transformations.

Theorem

Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$Ar(T(S)) = |\det A| Ar(S).$$

If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^2$, then

$$Vol(T(S)) = |\det A| Vol(S).$$

Summary: What we did today...

- Matrix Determinants
  - Calculations, properties
- Cofactor Matrix
  - Properties
- Cramer's Rule
  - Inverse of a matrix, applications
- Linear Transformation
  - Area and volume
- Next time
  - Vector Spaces

Topic Seven
Vector Spaces[a]

[a](Lay et al., 2015, Sections 4.1-4.5)

Lecture Contents

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn the definition of a vector spaces and linear transformation
2. Work with examples of vector spaces and vector subspaces
3. Find null spaces, column spaces of matrices
4. Understand the concepts of linearly independent sets, bases, and dimension

Introduction

- So far we have been learning how to work with vectors in $\mathbb{R}^n$ algebraically and geometrically
- Vector spaces are generalisations of the ideas of vectors in $\mathbb{R}^n$
- Recall the vectors in $\mathbb{R}^n$ can be added together and multiplied by a scalar
- These ideas are made abstarct in the definition of a vector space
- This enables us to apply the results of linear algebra to a wider variety of cases other than $\mathbb{R}^n$
- And also show that many examples of vector spaces are in fact spaces similar to $\mathbb{R}^n$

Definition

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and for all scalars c and d.

- The sum of $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted by $\boldsymbol{u} + \boldsymbol{v}$, is in V
- $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ and $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
- There is a zero vector 0 in V such that $\boldsymbol{u} + 0 = \boldsymbol{u}$
- For each $\boldsymbol{u}$ in V, there is a vector $-\boldsymbol{u}$ in V such that $\boldsymbol{u} + -\boldsymbol{u} = 0$
- The scalar multiple of $\boldsymbol{u}$ by c, denoted by $c\boldsymbol{u}$, is in V
- $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ and $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$
- $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ and $1\boldsymbol{u} = \boldsymbol{u}$

Examples

The geometric intuition developed for $\mathbb{R}^3$ will help you understand and visualize many concepts.

1. The spaces $\mathbb{R}^n$, where $n \geq 1$, are the premier examples of vector spaces
2. The set of $n \times m$ matrices under matrix addition and scalar multiplication is a vector space, which is similar to $\mathbb{R}^{nm}$
3. For $n \geq 0$, the set $\mathcal{P}^n$ of polynomials of degree at most n consists of all polynomials of the form

$$\boldsymbol{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

4. Let V be the set of all real-valued functions defined on a set X and under addition and multiplication pointwise

In some cases a suitable subset of a vector space satisfy the properties of a vector space themselves. In these cases we say that the subset is a vector subspace of the larger vector space.

Definition

A subspace of a vector space V is a subset H of V that has three properties below.

1. The zero vector of V is in H
2. H is closed under vector addition. That is, for each $\boldsymbol{u}$ and $\boldsymbol{v}$ in H, the sum $\boldsymbol{u} + \boldsymbol{v}$ is in H
3. H is closed under multiplication by scalars. That is, for each $\boldsymbol{u}$ in H and each scalar c, the vector $c\boldsymbol{u}$ is in H

Examples

1. The set consisting of only the zero vector in a vector space V is a subspace of V, called the zero subspace
2. For $n \geq 0$, the set $\mathcal{P}^n$ of polynomials of degree at most n when considered as functions on $\mathbb{R}$ is a subspace of all function $f : \mathbb{R} \longrightarrow \mathbb{R}$
3. The vectors space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, however vectors of the form

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}$$

   is a subspace of $\mathbb{R}^3$ which looks like $\mathbb{R}^2$
4. The set of upper triangular $n \times n$ matrices is a subspace of all $n \times n$ matrices

Examples

1. Given two vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a vector space V, then the set of linear combination of these vectors $H = \text{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is a subspace of V
2. More generally if $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_p \in V$, then $H = \text{Span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ is a subspace of V
3. Let
$$H = \left\{ \begin{bmatrix} x_1 + ax_2 \\ x_2 + x_3 \\ 0 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$
Then H is a subspace of $\mathbb{R}^3$

- There are some set ways that subspaces can arise
- For example, as the set of all solutions to a system of homogeneous linear equations
- Or the set of all linear combinations of certain specified vectors

Definition

The null space of an $m \times n$ matrix A, written as Nul$A$, is the set of all solutions of the homogeneous equation $A\boldsymbol{x} = 0$. In set notation, we have

$$\text{Nul}A = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = 0\}$$

Example

Determine if $\boldsymbol{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ is in Nul$A$ and find Nul$A$ where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 2 & 0 \end{bmatrix}.$$

Theorem

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\boldsymbol{x} = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Exercise

Find a spanning set for the null space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}.$$

Column Spaces

Definition

The column space of an $m \times n$ matrix A, written as Col$A$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_n \end{bmatrix}$, then $\text{Col}A = \text{Span}\{\boldsymbol{a}_1, ..., \boldsymbol{a}_n\}$.

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Exercise

Find if $\boldsymbol{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the column space of A

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 \\ 1 & -2 & 2 & 3 \\ 2 & -4 & 5 & 8 \end{bmatrix}.$$

Definition

A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\boldsymbol{x} \in V$ a unique vector $T(\boldsymbol{x}) \in W$, such that

1. $T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$
2. $T(c\boldsymbol{u}) = cT(\boldsymbol{u})$

for all $\boldsymbol{u}, \boldsymbol{v} \in V$ and c a scalar.

- The kernel (or null space) of such a T is the set of all $\boldsymbol{u}$ in V such that $T(\boldsymbol{u}) = 0$ (the zero vector in W)
- The range of T is the set of all vectors in W of the form $T(\boldsymbol{x})$ for some x in V

Linear independence

The concepts of linear independence and span both are valid in the context of vector spaces.

Definition

An indexed set of vectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_p$ in V is said to be linearly independent if the vector equation

$$c_1\boldsymbol{v}_1 + \cdots + c_p\boldsymbol{v}_p = 0$$

has only the trivial solution, $c_1 = \cdots = c_p = 0$, otherwise the set is said to be linearly dependant.

Example

- Let $\boldsymbol{p}_1(t) = 1$, $\boldsymbol{p}_2(t) = t$, and $\boldsymbol{p}_3(t) = 4 - t$. Then $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$ are linearly dependant
- Let $\boldsymbol{p}_1(t) = t^2$, $\boldsymbol{p}_2(t) = t$, and $\boldsymbol{p}_3(t) = 4 - t$. Then $\{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\}$ are linearly independent

Definition

Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ in V is a basis for H if

1. B is a linearly independent set, and
2. The subspace spanned by B coincides with H, i.e.,

$$H = \text{Span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$$

1. Let $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ be the columns of the $n \times n$ identity matrix, I_n. Then $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ is a bases for $\mathbb{R}^n$ known as the standard bases.
2. Let $\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}, \boldsymbol{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. Determine if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ forms a basis for $\mathbb{R}^3$.
3. Let $S = \{1, t, t^2, ..., t^n\}$. Verify that S is a basis for $\mathcal{P}^n$. This basis is called the standard basis for $\mathcal{P}^n$.

Theorem

Let $S = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ *be a set in* V, *and let* $H = \text{Span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$.

1. *If one of the vectors in* S*—say,* $\boldsymbol{v}_k$*—is a linear combination of the remaining vectors in* S*, then the set formed from* S *by removing* $\boldsymbol{v}_k$ *still spans* H
2. *If* $H \neq 0$*, some subset of* S *is a basis for* H

Theorem

Let $\mathcal{B} = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_p\}$ be a basis for a vector space V. Then for each $\boldsymbol{x} \in V$, there exists a unique set of scalars $c_1, ..., c_p$ such that

$$\boldsymbol{x} = c_1\boldsymbol{v}_1 + \cdots + c_p\boldsymbol{v}_p.$$

- We may write

$$[\boldsymbol{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

  when $\boldsymbol{x}$ is expressed in terms of bases $\mathcal{B}$, also known as the coordinate vector of $\boldsymbol{x}$ relative to $\mathcal{B}$
- This provides a one-to-one and onto mapping between V and $\mathbb{R}^p$
- Any finite-dimensional vector space looks like $\mathbb{R}^n$ for some n

Theorem

Let $\mathcal{B} = \{\boldsymbol{b}_1, ..., \boldsymbol{b}_n\}$ *and* $\mathcal{C} = \{\boldsymbol{c}_1, ..., \boldsymbol{c}_n\}$ *be bases of a vector space* V*. Then there is a unique* $n \times n$ *matrix* P *such that*

$$\left[\boldsymbol{x}\right]_{\mathcal{C}} = P\left[\boldsymbol{x}\right]_{\mathcal{B}}$$

The columns of P *are the* $\mathcal{C}$*-coordinate vectors of the vectors in the basis* $\mathcal{B}$*. That is,*

$$P = \left[\left[\boldsymbol{b}_1\right]_{\mathcal{C}} \quad \left[\boldsymbol{b}_2\right]_{\mathcal{C}} \cdots \left[\boldsymbol{b}_n\right]_{\mathcal{C}}\right].$$

Dimension

Theorem

If a vector space V has a basis $\mathcal{B} = \{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

This n is known as the dimension of V.

Definition

If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V. The dimension of the zero vector space is defined to be zero. If V is not spanned by a finite set, then V is said to be infinite-dimensional.

Theorem

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Theorem

Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

You may use **Colab** for Python the relevant library is SumPy.

```
import sympy
P=sympy.Matrix(
[[2, 1, 4, 5, 4],
[3, 5, 0, 0, 0],
[2, 1, 0, 0, 2]])
P.nullspace()
P.columnspace()
```

Summary: What we did today...

- Vector Spaces
  - Definition, examples
- Subspaces
  - Nul and Col spaces
- Bases
  - Linear independence, dimension
- Next time
  - Eigenvalues and Eigenvectors

Topic Eight
Eigenvalues and Eigenvectors[a]

[a](Lay et al., 2015, Sections 5.1-5.4)

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Lecture Contents

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn the properties of eigenvalues and eigenvectors
2. Calculate eigenvalues and eigenvectors of matrices
3. Compute the characteristic equation of a matrix
4. Learn about similar matrices and diagonalization
5. Use computer packages to compute eigenvalues and eigenvectors of matrices

- Recall the for a matrix A we have a mapping $\boldsymbol{x} \mapsto A\boldsymbol{x}$ for any vector $\boldsymbol{x}$
- There are times that the effects of A on a particular vector is rather simple
- For example it happens to be multiplication by a scalar, i.e., $A\boldsymbol{x} = \lambda\boldsymbol{x}$

Example

Consider $A = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$ with $\lambda = 3, -2$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $u = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Definition

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \lambda\boldsymbol{x}$; such an $\boldsymbol{x}$ is called an eigenvector corresponding to λ.

Example

Show that 2 is an eigenvalue for $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 3 & 1 \end{bmatrix}$. How about $-3, 3, -2$?

- Note λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation
$$(A - \lambda I)\,\boldsymbol{x} = 0$$
has a nontrivial solution
- The set of all solutions of above equation is just the null space of the matrix $A - \lambda I$
- This set is a subspace of $\mathbb{R}^n$ and is called the eigenspace of A corresponding to λ
- The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ

Example

Find the eigenspace corresponding to the eigenvalue of 2 in the matrix

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 6 \end{bmatrix}.$$

Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem

The matrix A has zero as an eigenvalue of and only if A is not invertible.

Theorem

If $v_1, ..., v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, ..., \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, ..., v_r\}$ is linearly independent.

The Characteristic Equation

- To find eigenvalues and eigenvectors we must find all scalars λ and vectors $\boldsymbol{x}$ so that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

- Alternatively, we must find all λ so that the equation

$$(A - \lambda I)\,\boldsymbol{x} = 0$$

  has a nontrivial solution
- Thus we need to find all λ so that $A - \lambda I$ is not invertible
- Equivalently we need all λ so that $\det(A - \lambda I) = 0$

Theorem

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if it satisfies the characteristic equation

$$\det(A - \lambda I) = 0$$

Example

Find the characteristic equation for the matrices given below.

$$\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Example

Find all eigenvalues and eigenvectors of $\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

Solution. To find eigenvalues solve the characteristic equation

$$\det\left(\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} - \lambda I\right) = \det\left(\begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix}\right)$$
$$= (4-\lambda)(1-\lambda) + 2 = 0$$

which gives $\lambda = 2, 3$. To find eigenspace corresponding to $\lambda = 2$ we find

$$\mathrm{Nul}\,(A - 2I) = \mathrm{Nul} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$$

and to find eigenspace corresponding to $\lambda = 3$ we find

$$\mathrm{Nul}\,(A - 3I) = \mathrm{Nul} \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

An eigenvectors corresponding to $\lambda = 2$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\lambda = 3$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Definition

Two matrices A and B are said to be similar if there exists an invertible matrix P such that $A = PBP^{-1}$ or equivalently if $B = P^{-1}AP$. The changing of A to $P^{-1}AP$ is called a similarity transformation.

Theorem

If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

- Two matrices can have the same eigenvalues but not be similar
- The eigenvalues may change under row operations

Eigenvectors and Difference Equations

- An application of eigenvectors is in the study of dynamical systems
- A stage-matrix model is a difference equation of the form

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k \text{ for } k = 0, 1, 2, \ldots$$

- A solution of the equation above is an explicit description of $\{\boldsymbol{x}_k\}$
- Whose formula for each $\boldsymbol{x}_k$ does not depend directly on A or on the preceding terms in the sequence other than the initial term $\boldsymbol{x}_0$
- The simplest solution is to take an eigenvector $\boldsymbol{x}_0$ corresponding to an eigenvalue λ and let

$$\boldsymbol{x}_k = \lambda^k \boldsymbol{x}_0$$

- Note if A is an $n \times n$ matrix with n linearly independent eigenvectors $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$
- Then any $\boldsymbol{x}_0 \in \mathbb{R}^n$ can be written as a linear combination of these eigenvectors

$$\boldsymbol{x}_0 = c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n$$

- Thus we have

$$\begin{aligned} A\boldsymbol{x}_0 &= A\left(c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n\right) \\ &= c_1\lambda_1\boldsymbol{v}_1 + \cdots + c_n\lambda_n\boldsymbol{v}_n \end{aligned}$$

- Iterating this procedure we find

$$\begin{aligned} \boldsymbol{x}_{k+1} = A\boldsymbol{x}_k &= A^k\boldsymbol{x}_0 \\ &= c_1\lambda_1^{k+1}\boldsymbol{v}_1 + \cdots + c_n\lambda_n^{k+1}\boldsymbol{v}_n \end{aligned}$$

- Therefore, the long term behaviour is typically along the eigenvectors with large eigenvalues

Example

Let $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. Analyse the long-term behaviour of the dynamical system defined by $\boldsymbol{x}_k = A\boldsymbol{x}_0$ with $\boldsymbol{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$.

Diagonalization

- It often happens that a matrix A is similar to a diagonal matrix D, i.e., we have $A = PDP^{-1}$
- When this happens we say that A is diagonalisable
- Diagonalization helps when we want to calculate large powers of A
- In that case we have

$$A^k = A^{k-2}PDP^{-1}PDP^{-1} = A^{k-2}PD^2P^{-1} = PD^kP^{-1}$$

- Now if $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ then we have

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Example

Diagonalise the following matrices, if possible.

$$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The general steps are as follows.

1. Find the eigenvalues of A
2. Find n linearly independent eigenvectors of A
3. Construct P from the vectors in step 2
4. Construct D from the corresponding eigenvalues

Theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Linear Transformation and Eigenvalues

- The diagonalization of a matrix $A = PDP^{-1}$ also helps with understanding effects of linear transformations
- We can show that transformation $\boldsymbol{x} \longmapsto A\boldsymbol{x}$ is essentially the same as the mapping $\boldsymbol{u} \longmapsto D\boldsymbol{u}$
- Recall that for a linear transformation T we can always find a matrix A so that $T(\boldsymbol{x}) = A\boldsymbol{x}$

Theorem

Suppose $A = PDP^{-1}$, where D is a diagonal $n \times n$ matrix. If $\mathcal{B}$ is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the $\mathcal{B}$-matrix for the transformation $\boldsymbol{x} \longmapsto A\boldsymbol{x}$.

You may use **Colab** for Python the relevant library is SumPy.

```
lamda = sympy.symbols('lamda')
N=sympy.Matrix(
[[0.8, 0.3],
[0.2, 0.7]])
N.eigenvals()
N.eigenvects()
N.charpoly(lamda)
N.diagonalize()
```

You may use **Colab** for Python the relevant library is NumPy.

```
import numpy
N=numpy.array(
[[0.8, 0.3],
[0.2, 0.7]])
numpy.linalg.eig(N)
```

You may use **RStudio Cloud** for R. The relevant libraries is matlib, but basic R functions work well in this case too.

```
library(matlib)
C <- matrix(c(1,2,3,
2,5,6,
3,6,10), 3, 3)
eigen(C) # base
Eigen(C) # matlib
```

Summary: What we did today...

- Eigenvalues and Eigenvectors
  - Definition, examples
- The Characteristic Equation
  - Similarity
- Diagonalization
  - $A = PDP^{-1}$
- Linear Transformation
  - Eigenvalues
- Next time
  - Inner Product, Length, and Orthogonality

Topic Nine
Inner Product, Length, and Orthogonality[a]

[a] (Lay et al., 2015, Sections 6.1-6.2)

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_i u_i v_i$$

Lecture Contents

Intended Learning Outcomes

By the end of this session you will be able to...

1. Learn the properties of inner product
2. Calculate length of vectors and distance between two vectors
3. Understand the concept of orthogonality and orthogonal sets
4. Use computer packages to compute inner product and distance

- Recall vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$ have lengths
- The concept of distance and length can also be introduced in $\mathbb{R}^n$
- They provide powerful tools for solving many applied problems

Definition

For two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, which we can regard as $n \times 1$ matrices, the inner product or dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is given by

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Example

Compute the inner product of $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}$.

Theorem

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$. *Then the following holds.*

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
2. $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$
3. $(c\boldsymbol{u}) \cdot \boldsymbol{w} = c\,(\boldsymbol{u} \cdot \boldsymbol{w}) = \boldsymbol{u} \cdot (c\boldsymbol{w})$
4. $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ *and* $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ *if and only if* $\boldsymbol{u} = 0$

Length of a Vector

Definition

Let $\boldsymbol{u} \in \mathbb{R}^n$. Then the length (or norm) of $\boldsymbol{u}$ is the nonnegative scalar $\|\boldsymbol{u}\|$ defined by

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{u_1 u_1 + u_2 u_2 + \cdots + u_n u_n}.$$

Example

Calculate the length of the vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Unit Vectors

- A vector $\boldsymbol{u}$ is called a unit vector if $\|\boldsymbol{u}\| = 1$
- For any vector $\boldsymbol{u}$, we can create a unit vector $\widehat{\boldsymbol{u}}$, which is in the same direction as $\boldsymbol{u}$ but with length 1
- The vector $\widehat{\boldsymbol{u}}$ is called the normalised vector of $\boldsymbol{u}$ and is given by

$$\widehat{\boldsymbol{u}} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|}$$

Example

Normalise the vectors $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{4} \end{bmatrix}$.

Definition

For two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ the distance between $\boldsymbol{u}, \boldsymbol{v}$ is length of the vector $\boldsymbol{u} - \boldsymbol{v}$ that is

$$\operatorname{dist}(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|.$$

Example

Compute distance between $\boldsymbol{u} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -1 \end{bmatrix}$.

- Note for two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ we can write

$$\begin{aligned}
\operatorname{dist}(\boldsymbol{u}, \boldsymbol{v})^2 = \|\boldsymbol{u} - \boldsymbol{v}\|^2 &= (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v}) \\
&= \boldsymbol{u} \cdot \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{u} + \boldsymbol{v} \cdot \boldsymbol{v} \\
&= \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - 2\boldsymbol{u} \cdot \boldsymbol{v}
\end{aligned}$$

- Two vectors are called orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$
- Note that by above $\boldsymbol{u}$ and $\boldsymbol{v} = 0$ if and only if $\|\boldsymbol{u} - \boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$
- The above is know as the The Pythagorean Theorem, which you are familiar with from the vectors in $\mathbb{R}^2$

Example

Find the distance between $\boldsymbol{u} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

- A vector $\boldsymbol{u} \in \mathbb{R}^n$ is said to be orthogonal to a subspace $W \subseteq \mathbb{R}^n$ if $\boldsymbol{u}$ is orthogonal to every vector of W
- The set of all vectors orthogonal to a subspace W of $\mathbb{R}^n$ is known as the orthogonal complement of W written as $W^\perp$
- A vector $\boldsymbol{x} \in W^\perp$ if and only if $\boldsymbol{x}$ is orthogonal to every vector of W
- It can be shown that $W^\perp$ is a subspace of $\mathbb{R}^n$

Theorem

Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T, so we have

$$(\mathrm{Row} A)^\perp = \mathrm{Nul} A \textit{ and } (\mathrm{Col} A)^\perp = \mathrm{Nul} A^T.$$

- Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$
- Then there connection between their inner product and the angle θ between the two line segments from the origin to the points identified with$\boldsymbol{u}$ and $\boldsymbol{v}$
- The formula is

$$\boldsymbol{u} \cdot \boldsymbol{v} = \|\boldsymbol{u}\| \|\boldsymbol{v}\| \cos \theta$$

- The formula for vectors in $\mathbb{R}^2$, can be verified through the law of cosines and considering the lengths $\|\boldsymbol{u}\|, \|\boldsymbol{v}\|, \|\boldsymbol{u} - \boldsymbol{v}\|$

Example

Find the angle between $\boldsymbol{u} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

- A set of vectors $\boldsymbol{u}_1, ..., \boldsymbol{u}_p$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0$ whenever $i \neq j$.

Example

Show that the set $\boldsymbol{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an orthogonal set.

Theorem

If $S = \{\boldsymbol{u}_1, ..., \boldsymbol{u}_p\}$ in $\mathbb{R}^n$ is an orthogonal set of nonzero vectors, then S is linearly independent and hence is a basis for the subspace spanned by S.

You may use **Colab** for Python the relevant library is SumPy.

```
import sympy
a=sympy.Matrix(
[[1],
[2]])
b=sympy.Matrix(
[[3],
[4]])
a.T*b
```

You may use **Colab** for Python the relevant library is NumPy.

```
import numpy
a = numpy.array([1,2,3])
b = numpy.array([0,1,0])
numpy.inner(a, b)
```

You may use **RStudio Cloud** for R. The relevant libraries is matlib, but basic R functions work well in this case too.

```
a <- c(1, 2, 4, 5)
b <- c(-1, 2, 3, 4)
t(a)%*%b
sum(a*b)
```

Summary: What we did today...

- **Inner Product and Length** — Definition, examples
- **Distance Between Vectors** — Orthogonal Vectors, The Pythagorean Theorem
- **Orthogonal Complements** — Angles in $\mathbb{R}^2$ and $\mathbb{R}^3$
- **Orthogonal Set** — Bases
- **Next time** — There will not be any
- **Have a Good Holiday** — See you next term

See You Next Time

Do Not Forget To

- Ask any **questions** now or through my contact details.
- Drop me **comments** and **feedback** relating to any aspects of the course.
- Come and see me during Student Drop-in Hours: TUESDAYS 15:00-16:00 (QM315/TEAMS). Alternatively, email to make an appointment on Teams.

Thank You!